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Another Majorana Idea: Real and Imaginary in the Weinberg Theory

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The Majorana concept of neutrality is applied to the solutions of $j = 1$ Weinberg equations in the $(j, 0) \oplus (0, j)$ representation of the Poincaré group.

It is well known that the Dirac equation can be separated in a relativistic invariant way into a real part and an imaginary part (Majorana, 1937). In Hamiltonian form the real Dirac equation reads

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \text{grad}) + \beta'\mu\right]\mathcal{U} = 0 \tag{1}$$

where the Dirac matrices are chosen to be

$$\alpha_x = \rho_1\sigma_x; \qquad \alpha_y = \rho_3; \qquad \alpha_z = \rho_1\sigma_z; \qquad \beta' = -i\beta = i\rho_1\sigma_y \tag{2}$$

and the mass term is $\mu = mc/\hbar$. In the configuration space the anticommutation rules are [Majorana (1937), equation (12)]

$$\mathcal{U}_i(q)\mathcal{U}_k(q') + \mathcal{U}_k(q')\mathcal{U}_i(q) = \tfrac{1}{2}\delta_{ik}\delta(q - q') \tag{3}$$

The Hamiltonian operator is then defined as [Majorana (1937), equation (13)]

$$H = \int \mathcal{U}^\dagger[-c(\alpha, p) - \beta mc^2]\mathcal{U}\, dq \tag{4}$$

According to Majorana, "in the present state of our knowledge equations (12) and (13) constitute the simplest theoretical representation of a system of neutral particles." Recent experimental manifestations of the mass term

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of the neutrino (Boris *et al.*, 1987; see Gelmini and Roulet, 1995, for review) and of neutrino oscillations [see Athanassopoulos *et al.* (1995) and for an alternative analysis Hill (1995)] suggest that one should look for a theoretical formalism which could account for a possible mass term and in the massless limit could lead to the Weyl scheme (McLennan, 1957; Case, 1957; Barut and Ziino, 1993; Ziino, 1996; Ahluwalia, 1996; Dvoeglazov, 1995a–c) and reproduce correct predictions of the standard model.

On the other hand, recent analysis of experimental data (Bolotov *et al.*, 1990; Akimenko *et al.*, 1991) on the decays of π^- and K^+ mesons indicates the necessity of introducing tensor interactions in the theoretical models. This suggests that we should correct our understanding of the nature of the minimal coupling and pay attention to other types of Lorentz-invariant interaction structures between the spinor and higher spin fields. In the sixties a formalism was proposed (Weinberg, 1964a,b, 1969) for arbitrary spin-j particles, which is on an equal footing with the Dirac formalism in the $(1/2, 0) \oplus (0, 1/2)$ representation. It allows for other forms of Lorentz-invariant couplings (Weinberg, 1964a, §7; Dvoeglazov, 1993, 1994a). Moreover, it has no problem of indefinite metric. Interest in this description has considerably increased due to the construction of an explicit example of a theory of Wigner type (Ahluwalia *et al.*, 1993)[2] and to the necessity for a detailed interpretation of the $E = 0$ solution of the Maxwell equations (Majorana, 1928–1932, as cited by Mignani *et al.*, 1974; Oppenheimer, 1931; see also Gianetto, 1985; Ahluwalia and Ernst, 1992; Evans and Vigier, 1994, 1995; Evans *et al.*, 1996; Dvoeglazov *et al.*, 1994; Dvoeglazov, 1994b).

In Ahluwalia (1996) it is proved that one cannot build self/anti-self charge conjugate "spinors" in the $(1, 0) \oplus (0, 1)$ representation. The equation $S^c_{[1]}\psi = e^{i\alpha}\psi$ has no solutions in the field of complex numbers. The $\Gamma^5 S^c_{[1]}$ self/anti-self conjugate "spinors" were introduced there. So we have to look at alternative ways for describing $j = 1$ neutral quantum fields. The aim of this paper is to apply the above-mentioned Majorana idea of neutrality to the $j = 1$ states of the $(1, 0) \oplus (0, 1)$ representation.

In the generalized canonical (standard) representation the Barut–Muzinich–Williams matrices are expressed as

$$\gamma_{00}^{CR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_{i0}^{CR} = \gamma_{0i}^{CR} = \begin{pmatrix} 0 & -J_i \\ J_i & 0 \end{pmatrix} \tag{5a}$$

$$\gamma_{ij}^{CR} = \gamma_{ji}^{CR} = \begin{pmatrix} \eta_{ij} + \{J_i, J_j\} & 0 \\ 0 & -\eta_{ij} - \{J_i, J_j\} \end{pmatrix} \tag{5b}$$

[2] This paper presents an explicit example of a quantum field theory of Bargmann–Wightman–Wigner type (Wigner, 1965).

Here J_i, $i, j = 1, 2, 3$, are the $j = 1$ matrices and $\eta_{\mu\nu}$ is the flat space-time metric. We work in the isotropic basis in which the spin matrices read

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6}$$

By using the Wigner time-reversal operator ($\Theta_{[j]}\mathbf{J}\Theta_{[j]}^{-1} = -\mathbf{J}^*$)

$$\Theta_{[j=1]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{7}$$

one can apply the Majorana procedure to transfer over to the representation where all $\gamma_{\mu\nu}$ matrices are the *real* matrices. The unitary matrix ($U^\dagger U = UU^\dagger = 1$) for this procedure is

$$U = \frac{1}{2\sqrt{2}}\begin{pmatrix} (1-i) + (1+i)\Theta & -(1-i) + (1+i)\Theta \\ (1+i) + (1-i)\Theta & -(1+i) + (1-i)\Theta \end{pmatrix} \tag{8a}$$

$$U^\dagger = \frac{1}{2\sqrt{2}}\begin{pmatrix} (1+i) + (1-i)\Theta & (1-i) + (1+i)\Theta \\ -(1+i) + (1-i)\Theta & -(1-i) + (1+i)\Theta \end{pmatrix} \tag{8b}$$

As a result we arrive at ($\gamma_{\mu\nu}^{MR} = U\gamma_{\mu\nu}^{CR}U^\dagger$)

$$\gamma_{00}^{MR} = \begin{pmatrix} 0 & \Theta \\ \Theta & 0 \end{pmatrix},$$

$$\gamma_{01}^{MR} = \gamma_{10}^{MR} = \begin{pmatrix} 0 & -J_1\Theta \\ -J_1\Theta & 0 \end{pmatrix} \tag{9a}$$

$$\gamma_{02}^{MR} = \gamma_{20}^{MR} = \begin{pmatrix} iJ_2\Theta & 0 \\ 0 & -iJ_2\Theta \end{pmatrix},$$

$$\gamma_{03}^{MR} = \gamma_{30}^{MR} = \begin{pmatrix} 0 & -J_3\Theta \\ -J_3\Theta & 0 \end{pmatrix} \tag{9b}$$

$$\gamma_{ij}^{MR} = \gamma_{ji}^{MR} = \frac{1}{2}\begin{pmatrix} i(J_{ij}^* - J_{ij})\Theta & (J_{ij}^* + J_{ij})\Theta \\ (J_{ij}^* + J_{ij})\Theta & -i(J_{ij}^* - J_{ij})\Theta \end{pmatrix},$$

$$\gamma_5^{MR} = \begin{pmatrix} 0 & i1 \\ -i1 & 0 \end{pmatrix} \tag{9c}$$

Here we introduced the notation $J_{ij} = \eta_{ij} + \{J_i, J_j\}$. Since J_2 is the only one (of the J_i) matrices which is imaginary in the isotropic basis, we can conclude that a set of real Barut–Muzinich–Williams matrices is constructed. The $(1, 0) \oplus (0, 1)$ functions in this representation are defined by

$$u^{\mathrm{MR}}(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} \phi_L + \Theta\phi_R \\ \phi_L + \Theta\phi_R \end{pmatrix} + \frac{i}{2}\begin{pmatrix} -\phi_L + \Theta\phi_R \\ \phi_L - \Theta\phi_R \end{pmatrix} = \mathcal{U}^+ + i\mathcal{V}^+ \tag{10a}$$

$$v^{\mathrm{MR}}(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} -\phi_L + \Theta\phi_R \\ -\phi_L + \Theta\phi_R \end{pmatrix} + \frac{i}{2}\begin{pmatrix} \phi_L + \Theta\phi_R \\ -\phi_L - \Theta\phi_R \end{pmatrix} = \mathcal{U}^- + i\mathcal{V}^- \tag{10b}$$

One can see that

$$v^{\mathrm{MR}}(p^\mu) = \gamma_5^{\mathrm{MR}} u^{\mathrm{MR}}(p^\mu) = i\gamma_5^{\mathrm{WR}}\gamma_0^{\mathrm{WR}} u^{\mathrm{MR}}(p^\mu) = \begin{pmatrix} 0 & i1 \\ -i1 & 0 \end{pmatrix} u^{\mathrm{MR}}(p^\mu) \tag{11}$$

The index CR stands for the canonical representation, WR for the Weyl representation, and MR for the Majorana representation. For the second-type spinors (Ahluwalia *et al.*, 1996; Dvoeglazov, 1995a–c) $\lambda^{\mathrm{S,A}}$ and $\rho^{\mathrm{S,A}}$ in both the $j = 1/2$ and $j = 1$ cases the use of the Majorana representation leads to a natural separation into real and imaginary parts when referring to positive (negative) solutions.

The real and imaginary parts of the positive-energy u-"bispinors" of helicity ± 1 are (compare with the $j = 1/2$ case; see Appendix)

$$\mathcal{U}_\uparrow^+(p^\mu) = \mathcal{U}_\downarrow^+(p^\mu) = \frac{1}{2\sqrt{2}}\begin{pmatrix} p^- - \dfrac{p_2(p_1 + p_2)}{E + m} \\ -\sqrt{2}\left(p_1 - \dfrac{p_2 p_3}{E + m}\right) \\ p^+ + \dfrac{p_2(p_1 - p_2)}{E + m} \\ p^- + \dfrac{p_2(p_1 - p_2)}{E + m} \\ -\sqrt{2}\left(p_1 + \dfrac{p_2 p_3}{E + m}\right) \\ p^+ - \dfrac{p_2(p_1 + p_2)}{E + m} \end{pmatrix} \tag{12a}$$

$$\mathcal{V}^{+}_{\uparrow}(p^{\mu}) = -\mathcal{V}^{+}_{\downarrow}(p^{\mu}) = \frac{1}{2\sqrt{2}} \begin{pmatrix} -p^{-} + \dfrac{p_1(p_1 + p_2)}{E + m} \\ -\sqrt{2}\left(p_2 + \dfrac{p_1 p_3}{E + m}\right) \\ p^{+} - \dfrac{p_1(p_1 - p_2)}{E + m} \\ p^{-} - \dfrac{p_1(p_1 - p_2)}{E + m} \\ -\sqrt{2}\left(p_2 - \dfrac{p_1 p_3}{E + m}\right) \\ -p^{+} + \dfrac{p_1(p_1 + p_2)}{E + m} \end{pmatrix} \tag{12b}$$

Surprisingly, the real (and imaginary) parts of "bispinors" of different helicities appear to be equal to each other (within a sign). Thus, "bispinors" are connected by the operation of complex conjugation. As to the solution with $h = 0$, one has only the imaginary part of the positive-energy "bispinor":

$$\mathcal{U}^{+}_{\rightarrow}(p^{\mu}) \equiv 0, \qquad \mathcal{V}^{+}_{\rightarrow}(p^{\mu}) = \frac{1}{2} \begin{pmatrix} (p^{-} + m)\dfrac{p_1 + p_2}{E + m} \\ -\sqrt{2}\left(m + \dfrac{p_1^2 + p_2^2}{E + m}\right) \\ (p^{+} + m)\dfrac{p_1 - p_2}{E + m} \\ (p^{-} + m)\dfrac{p_2 - p_1}{E + m} \\ \sqrt{2}\left(m + \dfrac{p_1^2 + p_2^2}{E + m}\right) \\ -(p^{+} + m)\dfrac{p_1 + p_2}{E + m} \end{pmatrix} \tag{13}$$

The corresponding procedure can also be carried out for the negative-energy solutions; the "bispinors" are connected with (12a), (12b), (13) using equation (11). Unlike 'transversal' bispinors ($h = \pm 1$), the bispinor $v_{\rightarrow}(p^{\mu})$ has only a real part.

Finally, one cannot find a matrix which transfers over to the equation with pure imaginary matrices because the unit matrix commutes with all matrices of the unitary transformation.

In conclusion, using the standard form of the field operator in the x representation, one can separate out the real and imaginary parts of the $(1, 0) \oplus (0, 1)$ coordinate-space "bispinors"; then the result can be compared with the case of the $(1/2, 0) \oplus (0, 1/2)$ representation. Relevant commutation relations can be found. However, if one wishes to obtain an entirely real coordinate-space equation (without the x-space imaginary part of the field function) such a procedure leads to certain constraints between components of the 4-vector momentum and/or constraints on the phase factors. The physical interpretation of the latter statement is unobvious and will be examined elsewhere.

APPENDIX

Here I present the explicit forms of $\mathcal{U}^{\pm}_{\uparrow\downarrow}$ and $\mathcal{V}^{\pm}_{\uparrow\downarrow}$, the real and imaginary parts of bispinors in the $j = 1/2$ Majorana representation. We can observe differences from the $j = 1$ case. The transfer matrix to the Majorana representation from the Weyl representation is given by

$$U = \frac{1}{2}\begin{pmatrix} 1 - i\Theta & 1 + i\Theta \\ -1 - i\Theta & 1 - i\Theta \end{pmatrix}, \qquad U^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 - i\Theta & -1 - i\Theta \\ 1 + i\Theta & 1 - i\Theta \end{pmatrix} \tag{A1}$$

The γ-matrices are given by

$$\gamma^0_{\mathrm{MR}} = \begin{pmatrix} 0 & -i\Theta_{[1/2]} \\ -i\Theta_{[1/2]} & 0 \end{pmatrix}, \qquad \gamma^1_{\mathrm{MR}} = \begin{pmatrix} -i\sigma_1\Theta_{[1/2]} & 0 \\ 0 & -i\sigma_1\Theta_{[1/2]} \end{pmatrix} \tag{A2a}$$

$$\gamma^2_{\mathrm{MR}} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^3_{\mathrm{MR}} = \begin{pmatrix} -i\sigma_3\Theta_{[1/2]} & 0 \\ 0 & -i\sigma_3\Theta_{[1/2]} \end{pmatrix} \tag{A2b}$$

$$\gamma^5_{\mathrm{MR}} = \begin{pmatrix} -i\Theta_{[1/2]} & 0 \\ 0 & i\Theta_{[1/2]} \end{pmatrix}, \qquad \Theta_{[j=1/2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A2c}$$

They are all imaginary and are related to equation (2),

$$\mathcal{U}^{+}_{\uparrow}(p^{\mu}) = \frac{1}{2\sqrt{(E + m)}}\begin{pmatrix} E + m - p_2 \\ 0 \\ -p_3 \\ -p_1 \end{pmatrix},$$

$$\mathcal{U}_{\downarrow}^{+}(p^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} 0 \\ E+m-p_2 \\ -p_1 \\ p_3 \end{pmatrix} \tag{A3a}$$

$$\mathcal{V}_{\uparrow}^{+}(p^{\mu}) = \gamma_5^{\mathrm{WR}}\gamma_0^{\mathrm{WR}}\mathcal{U}_{\downarrow}^{+}(\tilde{p}^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} p_1 \\ -p_3 \\ 0 \\ -E-m-p_2 \end{pmatrix} \tag{A3b}$$

$$\mathcal{V}_{\downarrow}^{+}(p^{\mu}) = -\gamma_5^{\mathrm{WR}}\gamma_0^{\mathrm{WR}}\mathcal{U}_{\uparrow}^{+}(\tilde{p}^{\mu}) = \frac{1}{2\sqrt{(E+m)}}\begin{pmatrix} -p_3 \\ -p_1 \\ E+m+p_2 \\ 0 \end{pmatrix} \tag{A3c}$$

The negative-energy spinors are related to the positive-energy ones by

$$v_{\uparrow}^{\mathrm{MR}}(p^{\mu}) = -i[u_{\downarrow}^{\mathrm{MR}}(p^{\mu})]^*, \qquad v_{\downarrow}^{\mathrm{MR}}(p^{\mu}) = +i[u_{\uparrow}^{\mathrm{MR}}(p^{\mu})]^* \tag{A4}$$

and thus

$$\mathcal{U}_{\uparrow}^{+} = \mathcal{V}_{\downarrow}^{-} = \mathrm{Re}\, u_{\uparrow}^{\mathrm{MR}} = \frac{u_{\uparrow}^{\mathrm{MR}} - iv_{\downarrow}^{\mathrm{MR}}}{2},$$

$$\mathcal{U}_{\downarrow}^{+} = -\mathcal{V}_{\uparrow}^{-} = \mathrm{Re}\, u_{\downarrow}^{\mathrm{MR}} = \frac{u_{\downarrow}^{\mathrm{MR}} + iv_{\uparrow}^{\mathrm{MR}}}{2} \tag{A5a}$$

$$\mathcal{V}_{\uparrow}^{+} = \mathcal{U}_{\downarrow}^{-} = \mathrm{Im}\, u_{\uparrow}^{\mathrm{MR}} = \frac{u_{\uparrow}^{\mathrm{MR}} + iv_{\downarrow}^{\mathrm{MR}}}{2i},$$

$$\mathcal{V}_{\downarrow}^{+} = -\mathcal{U}_{\uparrow}^{-} = \mathrm{Im}\, u_{\downarrow}^{\mathrm{MR}} = \frac{u_{\downarrow}^{\mathrm{MR}} - iv_{\uparrow}^{\mathrm{MR}}}{2i} \tag{A5b}$$

These formulas also can be used to form even and odd bispinors with respect to $\mathbf{p} \to -\mathbf{p}$.

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